Linearization of Circuits with Polynomial Nonlinearities Described in the Frequency Domain

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***Abstract*—For the design of electronic circuits, the computer simulation proved to be important to improve the expected final results. Several simulation methods are currently being studied to obtain more accurate results that are consistent with the actual behavior of the circuits and seek the reduction of computational complexity. For some nonlinear circuits under multi-tone stimulation, it is enough to consider the harmonics of only the tone with higher amplitude. First the large signal analysis of the nonlinear circuit is done using the Harmonic Balance (HB) method. Thus, the circuit is linearized around the HB results using a small signal analysis, that can be the Periodic AC Analysis (PAC), to later apply the superposition of these results. Since the traditional PAC performs this linearization assuming the nonlinearities in the time domain, this article aims to investigate how to apply the same analysis by treating a polynomial nonlinearity in the frequency domain.**

Keywords—Harmonic Balance, Periodic AC Analysis, Polynomial Model, Linearization.

I. INTRODUCTION

Computational simulation is an indispensable step for the design of electronic circuits nowadays because, with the advancement of mathematical and computational methods, it is possible to obtain accurate simulations that predict the behavior of the circuit before the manufacturing stage. Therefore, it becomes viable to analyze different parameters in the circuits such as the values of components or input variables, to obtain the desired final results.

For nonlinear circuits used in communication systems and also in power systems, the analysis of large signals is used to obtain the steady state response to these circuits containing periodic and independent sources of voltage or current in the time domain acting in *n* tones [1]. The Harmonic Balance (HB) method allows this analysis to be efficiently performed. Traditionally in HB, the nonlinearities of circuits are treated in the time domain, but polynomial nonlinearities of any order can also be treated in the frequency domain using HB [2]. For these two methods, it is considered that all independent sources are large to stimulate nonlinearities in the circuit, being possible to perform the analysis of large signals.

For cases where it is necessary to consider the influences of the harmonics of the tone with higher amplitude, disregarding the action of the harmonics of the tones with lower amplitudes, it is possible to linearize the circuit around one tone and apply superposition as a way to reduce the computational complexity of the simulations [3]. Periodic AC Analysis (PAC) performs this linearization starting from the results obtained with the analysis of large signals that is usually traditional HB, where the nonlinearities are treated in the time domain. The contribution of this article is to perform a similar process but always treating polynomial nonlinearities in the frequency domain.

II. HARMONIC BALANCE

The HB is a numerical method used for the analysis of nonlinear circuits because it reduces the computational complexity of simulations by computing directly the steady-state response. In this method, the circuit voltages and currents, represented by $x(t)$, can be written as the sum of sines and cosines plus a constant X_0, where X_{hS} and X_{hC} represent constant amplitudes, H is the number of considered harmonics in the analysis and ω_c is the fundamental angular frequency [4], as shown by the equation:

$$x(t) = X_0 + \sum_{h=1}^{H} [X_{hS} \sin(h\omega_c t) + X_{hC} \cos(h\omega_c t)]. \qquad (1)$$

In HB, each time-varying circuit unknown is transformed into $(2H+1)$ constant unknowns representing the sine and cosine amplitudes plus the constant X_0. Consequently, each equation of the circuit should be transformed into $(2H+1)$ equations that will be in the frequency domain. For dynamic elements, a square Jacobian matrix (Ω_{HB}) has to be considered too, because the derivatives in the characteristic equations of these elements should be treated in the frequency domain [5]. For polynomial elements, the nonlinearities can be treated in time domain or frequency domain.

A. Time Domain Analysis for Polynomial Nonlinearities

For nonlinear elements, the treatment in time domain is done by dividing the fundamental period into $(2H+1)$ equally spaced time intervals and:

I) multiplying a matrix F that converts the frequency to time domain containing the values of time-varying sines and cosines by a column vector X_{HB} with the constant X_0 and the amplitudes of the sines and cosines;

II) evaluating the nonlinearities in each one of these time intervals;

III) multiplying the inverse matrix (F^{-1}) that converts the time to frequency domain by a column vector resulted in II [5].

B. Frequency Domain Analysis for Polynomial Nonlinearities

The analyses reported in this section are valid only for polynomial nonlinearities and the case study considered a second-order polynomial nonlinearity, but a similar procedure could be applied to higher polynomial orders. Given a signal $x(t)$, the $x^2(t)$ can be represented by

$$y(t) = x^2(t) = f(t)^{\mathrm{T}} \times Y, \qquad (2)$$

where $f(t)^{\mathrm{T}}$ is the transpose column vector of time-varying sine and cosine values and Y can be represented by summations of column vectors that manipulate in a linear way the constant X_0 and the amplitudes of the sines and cosines of $x^2(t)$ and have their behavior predictable for any value of H [2].

III. PERIODIC AC ANALYSIS

The Periodic AC Analysis (PAC) is a method that linearizes circuits having as an initial response the results obtained with a large signal analysis, using HB, for example [4]. In PAC, the nonlinear components of the circuit are linearized to obtain the analysis of small signals, and the final result is the superposition between the large signal and the small signal analysis.

In general, the linearization of the circuit is done by deriving the nonlinear equation with respect to the unknown to be analyzed [6], as the following equation shows:

$$f_{Lin}(X) = \frac{df_{NL}(X_0)}{dX} \cdot (X - X_0), \qquad (3)$$

where $f_{Lin}(X)$ is the resulted linearized equation, $f_{NL}(X)$ is the nonlinear equation concerning the unknown $X(t)$ and $X_0(t)$ is the initial response obtained by the large signals analysis [6]. The result of the linearization can be written as:

$$f_{Lin}(X) = g(t) \times \begin{bmatrix} 1 \\ \sin(\omega_c t) \\ \cos(\omega_c t) \\ \sin(2\omega_c t) \\ \vdots \\ \cos(H\omega_c t) \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix} \times \begin{bmatrix} 1 \\ \sin(\omega_c t) \\ \cos(\omega_c t) \\ \cos(\omega_c t) \\ \vdots \\ \cos(H\omega_c t) \end{bmatrix}, \qquad (4)$$

where $g(t)$ is a vector of conductance amplitudes. After trigonometric manipulations to establish a relation between g and the functions of sines and cosines varying in time and considering that the source of small signals has an angular frequency ω_2, $f_{Lin}(X)$ can be written as:

$$f_{Lin}(X) = COND \times V_{SS}, \qquad (5)$$

where *COND* is a conductance matrix defined by

$$COND = \begin{bmatrix} g_0 & 0 & \frac{g_2}{2} & \frac{-g_1}{2} & \frac{g_4}{2} & \frac{-g_3}{2} & \cdots & \frac{-g_{k-3}}{2} \\ 0 & g_0 & \frac{g_1}{2} & \frac{g_2}{2} & \frac{-g_3}{2} & \frac{g_4}{2} & \cdots & \frac{g_{k-2}}{2} \\ \frac{g_2}{2} & \frac{g_1}{2} & g_0 & 0 & \frac{g_2}{2} & \frac{-g_1}{2} & \cdots & \frac{-g_{k-5}}{2} \\ \frac{-g_1}{2} & \frac{g_2}{2} & 0 & g_0 & \frac{g_1}{2} & \frac{g_2}{2} & \cdots & \frac{g_{k-4}}{2} \\ \frac{g_4}{2} & \frac{g_3}{2} & \frac{g_2}{2} & \frac{g_1}{2} & g_0 & 0 & \cdots & \frac{-g_{k-7}}{2} \\ \frac{-g_3}{2} & \frac{g_4}{2} & \frac{-g_1}{2} & \frac{g_2}{2} & 0 & g_0 & \cdots & \frac{g_{k-8}}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{-g_{k-3}}{2} & \frac{-g_{k-2}}{2} & \frac{-g_{k-5}}{2} & \frac{-g_{k-4}}{2} & \frac{-g_{k-7}}{2} & \frac{-g_{k-8}}{2} & 0 & g_0 \end{bmatrix}, \qquad (6)$$

and V_{ss} is the linearized element to be analyzed in small signals [7] defined by

$$V_{SS} = \begin{bmatrix} v_{k-2} \\ v_{k-1} \\ \vdots \\ v_4 \\ v_5 \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{k-4} \\ v_{k-3} \end{bmatrix} \times \begin{bmatrix} \sin[(\omega_2 - H\omega_c)t] \\ \cos[(\omega_2 - H\omega_c)t] \\ \vdots \\ \sin[(\omega_2 - 1\omega_c)t] \\ \cos[(\omega_2 - 1\omega_c)t] \\ \sin(\omega_2 t) \\ \cos(\omega_2 t) \\ \sin[(\omega_2 + 1\omega_c)t] \\ \cos[(\omega_2 + 1\omega_c)t] \\ \vdots \\ \sin[(\omega_2 + H\omega_c)t] \\ \cos[(\omega_2 + H\omega_c)t] \end{bmatrix}, \qquad (7)$$

where $k = 2(2H+1)$. In (5), no elements vary in time and *COND* is the matrix with only constant numbers that depend on the constant amplitudes of g.

The equations shown previously in this section are valid for time and frequency domain linearization. The difference between the two analyses is the way the vector g is calculated as will be shown in the next two subsections.

A. Time Domain Linearization for Polynomial Nonlinearities

To obtain the elements of the vector g, the transformation matrices F and F^{-1}, containing the values of time-varying sines and cosines obtained from the HB, are used. For any nonlinear function f_{NL} of the circuit, the elements of the vector g are equal to:

$$g_{Time} = F^{-1}\left\{\frac{d\left[f_{NL}(F.V_{Time})\right]}{dv}\right\}, \qquad (8)$$

where $v = F.V_{Time}$ and V_{Time} is the vector with the resulted amplitudes of sines and cosines using HB with polynomial nonlinearities treated in time domain.

B. Frequency Domain Linearization for Polynomial Nonlinearities

To obtain the elements of the vector g, the transformation matrices F and F^{-1} are not used. The sines and cosines amplitudes are directly used to obtain the elements of the vector g. Specifically for $f_{NL}(v) = v^2$, the elements of the matrix g are:

$$g_{Freq.} = 2V_{Freq.}, \qquad (9)$$

and $V_{Freq.}$ is the vector with the resulted amplitudes of sines and cosines using HB with polynomial nonlinearities treated in frequency domain.

IV. SIMULATION RESULTS

The circuit used for the linearization in time and frequency domain is represented in Figure 1 and contains a saturable inductor (L) that represents a passive and nonlinear component. Duo to the nonlinearity considered in the analysis is a second-order polynomial, the voltage in the inductor as a function of time will be equal to the squared variation of the current I_L. The circuit also has a resistor of 377 Ω and an independent sinusoidal voltage source (v) that has a peak voltage equal to 440 V, a fundamental frequency equal to 60 Hz, and an angle discrepancy equal to 0° for the large signal analysis. For the small signal analysis, the equation of the voltage in the inductor was linearized and it was considered a sinusoidal voltage source with peak voltage equal to 5 V and a frequency equal to 120 Hz to stimulate in the circuit the second harmonic.

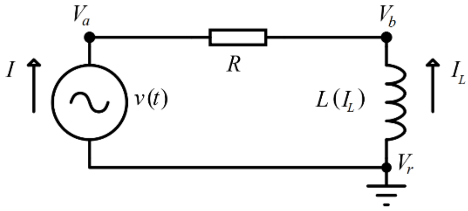

Fig. 1. The circuit with a polynomial nonlinearity

To find the equations that describe the circuit behavior, the Modified Nodal Analysis (MNA) was used having as unknowns the nodal voltages V_a, V_b, V_r and the currents of the circuit. The system of equations was solved by MATLAB® using the *fsolve* routine. Since V_r is equal to 0 V, because it is a reference voltage, the following system of equations can be obtained:

$$\begin{cases} I - \dfrac{V_a - V_b}{377} = 0 \\ \dfrac{V_a - V_b}{377} + I_L = 0 \\ V_b - \dfrac{d(I_L^2)}{dt} = 0 \\ V_a - 440\sin(2\pi 60 t) = 0 \end{cases} . \qquad (10)$$

A. Large Signal Analysis

For this section, 6 harmonics ($H = 6$) will be considered for time and frequency domain analysis in HB, because by increasing the value of H the methods are expected to be more accurate since normally the values of the amplitudes decrease as the order of the harmonic increases [4]. For $H = 6$, the matrix F will be a square matrix of order ($2H$+1 = 13) and each time-varying unknown in the circuit is transformed into 13 unknowns. The system of equations in (10) has now 52 unknowns and 52 equations.

In the time domain analysis, the voltage in the inductor is defined by

$$V_{L_{Time}} = \Omega_{HB}.F^{-1}(F.X_{I_L})^2, \qquad (11)$$

where X_{IL} is the vector containing the 13 unknowns that represents the constant amplitudes of sines and cosines which refer to the current I_L. For the frequency domain analysis, the voltage in the inductor is going to be

$$V_{L_{Freq}} = \Omega_{HB}.Y \qquad (12)$$

and Y is a column vector of 13 rows as a function of ω_c, the constant X_0, and the sine and cosine amplitudes representing I_L^2.

The result of the current I_L using the HB for time and frequency domain analysis and H equal to 6 is represented in Figure 2. The results are different because different forms of approximations in the calculations are inherent to the methods but show less discrepancy compared to the HB considering H equal to 2 or 4 [2]. For the treatment of nonlinearity in the frequency domain, an approximation occurs when disregarding the harmonics higher than H, so both I_L and I_L^2 will have the same number of harmonics. For the treatment of nonlinearity in the time domain, the approximation occurs when discretizing the time and using the matrices F, with sine and cosine varying in time, and inverse F [2].

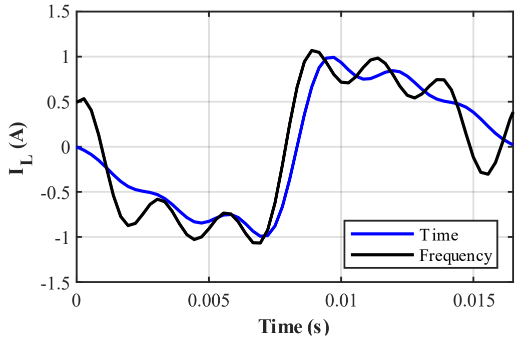

Fig. 2. HB results of the current I_L with $H = 6$

B. Small Signal Analysis and Superposition

For the small signal analysis, 3 harmonics ($H_L = 3$) will be considered. The linear analysis containing (H/2) harmonics results in the COND matrix being completely filled with values from g_0 to g_{12}, increasing the accuracy of the analysis. Each unknown of the system in (10) is transformed into $2(2H_L + 1)$ unknowns. The system now has 56 unknowns and 56 equations. The following equation represents the linearized voltage at the inductor and is valid for both methods:

$$V_{L_{PAC}} = \Omega_{PAC}.COND.X_{I_L}. \qquad (13)$$

X_{IL} contains the 14 unknowns representing I_L, COND has order 14x14 and is formed by the values of g, which will be different for the time and frequency method of linearization, and Ω_{PAC} is a Jacobian matrix obtained by deriving the vector with the sines and cosines in (7) with respect to time.

The vector g in the time domain linearization is equal to

$$g_{Time} = F^{-1}[2(FX_{HB_{Time}})], \qquad (14)$$

where $X_{HB_{Time}}$ is the 13 resulted amplitudes of sines and cosines using HB with polynomial nonlinearities treated in time domain. The current I_L using PAC analysis in time domain is represented in Figure 3. The linear waveform has an amplitude of 13.26 mA and the PAC result represents the superposition of the nonlinear and the linear analysis.

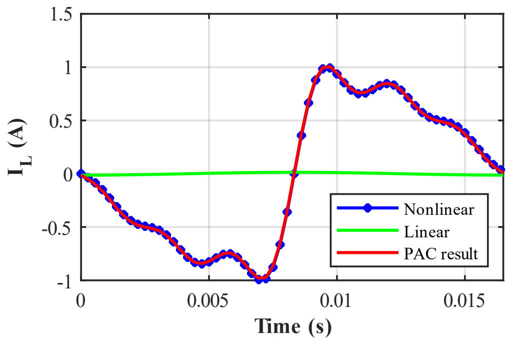

Fig. 3. PAC result in the time domain for I_L

For the frequency domain linearization, the g vector is equal to

$$g_{Freq.} = 2X_{HB_{Freq.}}. \qquad (15)$$

$X_{HB_{Freq.}}$ is the 13 resulted amplitudes of sines and cosines using HB with polynomial nonlinearities treated in frequency domain. The current I_L using PAC analysis in frequency domain is represented in Figure 4, where the linear waveform also has an amplitude of 13.26 mA and the PAC result represents the superposition of the nonlinear and the linear analysis.

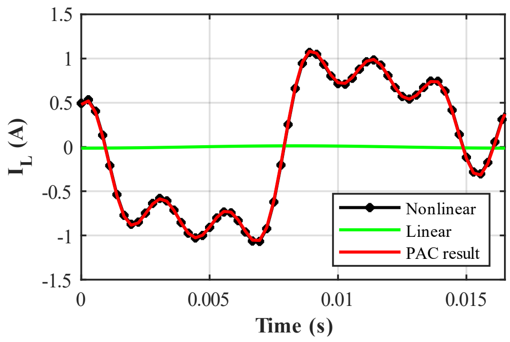

Fig. 4. PAC result in the frequency domain for I_L

In order to quantify the accuracy of the PAC in the time domain and the frequency domain separately, the following two analyses were compared for each method:

I) apply the HB in a circuit with the two tones at the same time, having one voltage source with a peak voltage equal to 440 V and the other with 5 V in series. The fundamental frequency remains 60 Hz and the small signal source of voltage remains at a frequency of 120 Hz to stimulate the second harmonic. In this analysis the equation for the inductor voltage remains nonlinear;

II) apply the PAC and separately analyze the circuit with the large signal source with a peak voltage of 440 V and the linearized circuit with the small signal source with a peak voltage of 5 V. The final results of the PAC are the superposition of these two analyses and are shown in Figures 3 and 4.

For the time domain analysis and $H = 6$, the Mean Square Error (MSE) between the HB with 2 tones and the PAC result is 1.0350×10^{-4} and the waveforms of these 2 analyses are represented in Figure 5.

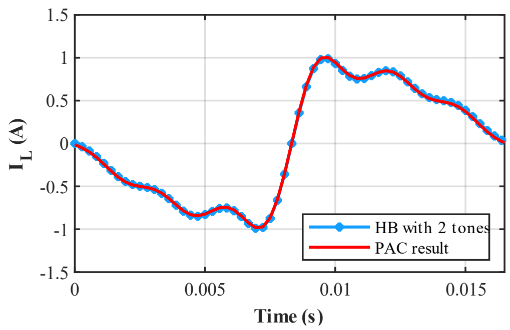

Fig. 5. I_L using HB with 2 tones and PAC in time domain

For the frequency domain and $H = 6$, the same comparison was made and is shown in Figure 6. The MSE between the PAC result and the HB considering 2 tones is 1.1903×10^{-4}.

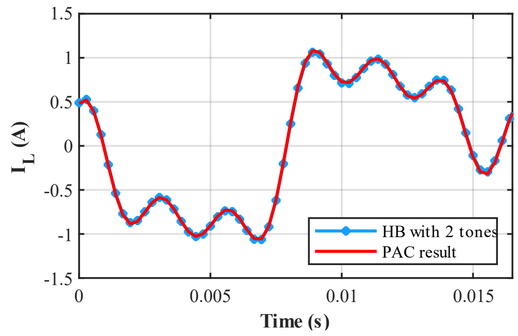

Fig. 6. I_L using HB with 2 tones and PAC in frequency domain

V. CONCLUSIONS

The article investigated how to apply PAC for second order polynomial nonlinearities in the frequency domain. This was done by changing the way the elements of the conductance matrix were acquired. For the time and the frequency domain, the same result for the current in the inductor of the linearized test circuit was obtained, even though the vector of conductance amplitudes was obtained in different ways. The final PAC result, which is the superposition of the linear and nonlinear analysis, changes because the nonlinear result using HB is different for the frequency domain and the time domain. Comparing the PAC result with the two tones HB, considering separately the time and frequency domain analysis, the lower MSE was obtained for the time domain, but both cases had errors to the fourth negative power of 10, which confirms that the results are similar. Therefore, the linearization using the PAC in the frequency domain for second order polynomial nonlinearities did not show significant differences compared to the linearization in the time domain.

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REFERENCES

[1] XU, J. ; NUNES, L. C. ; ROOT, D. E. ; PEDRO, J. C. Nonlinear Circuit Simulation and Modeling: Fundamentals for Microwave Design. Cambridge University Press, 2018.

[2] CORDEIRO, L. ; LIMA, E. G. Análise de equilíbrio harmônico com não linearidades polinomiais descritas no domínio da frequência. In: Seminários de Microeletrônica do Paraná, Curitiba. Anais do SeMicro-PR, 2021.

[3] VENDELIN, G. D. ; PAVIO, A. M. ; RUDOLPH, M. ; ROHDE, U. L. Microwave Circuit Design Using Linear and Nonlinear Techniques. Wiley, 2021.

[4] STEER, M. Microwave and RF Design: A Systems Approach. 2nd Edition. SciTech Publishing, 2013.

[5] NATALINO, C. G. ; ARAUJO, M. A. C. ; LIMA, E. G. Time and Frequency Domain Numerical Analyses for Electrical Circuits under Fixed Large Signal Stimulus and Variable Small Signal Sources. In: 19º Simpósio Brasileiro de Micro-ondas e Optoeletrônica e o 14º Congresso Brasileiro de Eletromagnetismo, 2020, Niterói. Anais da MOMAG 2020, p. 649-653.

[6] ARAUJO, M. A. C. ; MOURA, D. M. S. ; LIMA, E. G. Time and Frequency Domain Two-tone Simulation based on Periodic Steady State Linearization. In: 20th Microelectronics Student Forum, 2020.

[7] MOURA, D. M. S. ; NATALINO, C. G. ; LIMA, E. G. Linearização de Circuitos Eletrônicos em Torno de Regime Permanente a Zero, Um e Dois Tons. In: Seminários de Microeletrônica do Paraná, 2019, Curitiba. Anais do SeMicro-PR, 2019.